On the Probability Distribution of the Minimal Number of Hops Between any Pair of Nodes in a Bounded Wireless Ad-Hoc Network Subject to Fading

Sayandev Mukherjee Dan Avidor
Wireless Research Laboratory
Lucent Technologies – Bell Laboratories, New Jersey, USA
e-mail: {sayan,avidor}@lucent.com

Abstract— **We investigate an ad hoc network where node locations are distributed according to a homogeneous Poisson process with intensity λ_{n}. We assume that all the nodes are equipped with an identical wireless transceiver capable of operating satisfactorily up to a certain maximal link loss. Our link model depends on the length of the link and on random lognormal fading. Each node functions as a source and destination of data packets, and may also serve as a repeater to transport packets over multi-hop routes as determined by the network router. We focus on the probability distribution of the minimum number of hops between a source and a destination node known to be at distance D from the source. When the distribution of source-to-destination distances is known, the distribution of the minimal number of hops between any arbitrary pair of nodes can also be found. Many variations of this same problem have been studied in the literature. However, as far as we know, no exact closed-form analytic results for fading environments have been presented before.**

Keywords: mesh networking, multi-hop, outage probability

I. INTRODUCTION

We investigate a wireless ad hoc network of nodes distributed on a plane as points of a homogeneous Poisson process with intensity λ_{n}. We assume that all the nodes are equipped with an identical wireless transceiver capable of operating satisfactorily up to a certain maximal link loss. The link loss between any two nodes depends on the distance between the nodes, and on a random log-normal fading distributed identically and independently for each link. Each node functions as a source and destination of data packets, and also serves as a repeater to transport packets on their way to destination nodes under the control of the network router. We focus on the t-hop outage probability, viz., the probability that a source and destination node a distance D apart cannot communicate in $\leq t$ hops. It has been recognized by previous researchers of ad hoc networks that this probability has a significant impact on *route discovery*, throughput, and power consumption. When the distribution of inter-node distances is known, the t-hop probability between any arbitrary pair of nodes can also be found. Many variations and treatments of the problem have been published. However, as far as we know, no closed-form analytic results for fading environments have been presented before. We show in this paper that ignoring fading affects the results considerably.

II. RELATED WORK

Zorzi and Rao [1] propose a novel forwarding technique for an ad hoc network where, to save power, only a fraction of the nodes are switched ON at any given time. The scheme is applicable when the locations of the nodes are known. They evaluate the performance of the technique as measured by the mean number of hops to reach a destination node, as a function of the distance to the destination and the average number of available "neighbors." Through simulations and analytic bounds they derive the mean number of hops of the minimal hop-count route, and show that their technique comes close to that minimum. Like other researchers in this field, they do not account for fading, and assume that the radio range of all nodes is equal and fixed. Bettstetter and Eberspaecher [2] investigate the probability distribution of the minimal number of hops between a randomly-selected source and destination node, where all n nodes are uniformly distributed on a rectangular area of size $A = a \times b$, $b \leq a$. All the nodes have the same fixed transmission range r_0. They derive the probability that two randomly selected nodes on the square can communicate between themselves directly, i.e., are *1-hop connected* as a function of a, b, and r_0. They then derive the probability that two nodes on the square are *2-hop connected*, but not 1-hop connected, i,e, that the two nodes are further than r_0 apart, but at least one other node exists that is within r_0 from both nodes. Ignoring border effects (at the edges of the square), they provide the precise expression of the probability of that event. Bettstetter and Eberspaecher provide simulation results for some larger number of hops and analytic results for the case where the node density n/A increases without limit. Miller [3] studies a network of nodes randomly placed around a central point, say, $(0,0)$ with independent zero-mean Gaussian coordinates. For this case, and under the assumption that the *communication range* is fixed at R, he derives the probability of 2-hop connection between any pair of nodes. He then provides an upper bound to the probability of m-hop connection.

Chandler [4] considers a network with node locations given by points of a homogeneous Poisson process and derives an expression for the t-hop outage probability, assuming no fading. However, his derivation is somewhat intuitive and relies on an implicit assumption of independence (across arbitrary nodes) of the events that these nodes are exactly $t-1$ hops away from the source node and 1 hop away from the destination node. As we explain in Section VI, this assumption is not true and Chandler's expression is in fact a lower bound on the desired probability. In the present paper, we analyze the same model as Chandler, but in the presence of fading.

Mullen [5] studies the distribution of inter-node distance D in a wireless network occupying a rectangular region, using static location models as well as a version of the *random waypoint mobility model*. Averaging the above probability over this distribution yields the probability that a random pair of source and destination nodes cannot communicate in $\leq t$ hops.

This paper is organized as follows: following the definition of the node location process and the radio channel model, we first derive the distribution of the number of nodes having a direct connection to an arbitrary node, then use this distribution to derive a recursive relationship yielding a lower bound on the probability $q_t(D)$ that an arbitrary node cannot communicate with another node a distance D away in $\leq t$ hops. For $t = 2$, the result is exact. We then calculate the probability that an arbitrary pair of nodes cannot communicate in ≤ 2 hops when the network nodes are distributed uniformly over a finite region like a disk or a rectangle. We present specific examples and plot some analytic and simulation results before concluding the paper.

III. LOCATION MODEL FOR NODES

Define the disk with radius r centered at (x, y) by

$$B(x, y; r) = \left\{(x', y') : (x' - x)^2 + (y' - y)^2 \leq r^2\right\},$$

and the punctured disk obtained by deleting the center:

$$B'(x, y; r) = B(x, y; r) \setminus \{(x, y)\}.$$

We begin with the following model assumptions:

1. Nodes are points of a homogeneous Poisson process on the plane with intensity λ_{n}, i.e.,
   (i) The number of nodes in any finite region is a Poisson random variable with mean given by $\lambda_{\mathrm{n}} \times$ [the area of the region];
   (ii) The numbers of nodes in two disjoint finite regions are independent; and
   (iii) Conditioned on a given number of nodes in a chosen region, the locations of these nodes are uniformly distributed over that region.

Define, for any region A, $N_{\mathrm{n}}(A)$ to be the number of nodes in A. Then $N_{\mathrm{n}}(A) \sim \mathrm{Poiss}(\lambda_{\mathrm{n}} \times \mathrm{area}(A))$, i.e.,

$$\mathbb{P}\{N_{\mathrm{n}}(A) = n\} = \mathrm{e}^{-\lambda_{\mathrm{n}} \times \mathrm{area}(A)} \frac{[\lambda_{\mathrm{n}} \times \mathrm{area}(A)]^n}{n!},\ n \geq 0.$$

IV. RADIO PROPAGATION MODEL

We assume the following:

1. Attenuation with distance and shadow fading.
2. The shadow fade attenuations between all pairs of source and destination nodes are i.i.d. log-normal.
3. The shadow fade attenuation $10^{Z_{ij}/10}$ between any two nodes i and j is log-normally distributed and is the same regardless of which node is the transmitter and which the receiver. Thus we have
$$Z_{ij} \sim \mathcal{N}(0, \sigma^2) \text{ for all } i, j,$$
where σ is the same for all node pairs (i, j).
4. Node i has a connection to (i.e., is one hop away from) another node j if and only if the received power exceeds some given threshold $P_{\min}$, i.e., if and only if
$$K \frac{P_{\mathrm{T}}}{r^\delta} 10^{z_{ij}/10} > P_{\min} \Leftrightarrow r < r_{\mathrm{n}} \mathrm{e}^{h z_{ij}/\delta},\ h = \frac{\ln 10}{10}, \quad (1)$$
where δ is the distance-loss exponent, $10^{z_{ij}/10}$ is the shadow fade between nodes i and j, r is the distance between the nodes, K is a constant, taking into account parameters like antenna gain, antenna height (again assumed equal for all nodes), etc., and $r_{\mathrm{n}} = (P_{\mathrm{T}} K / P_{\min})^{1/\delta}$ is the range in the absence of fading.

V. DISTRIBUTION OF THE NUMBER OF NODES HAVING A DIRECT CONNECTION TO A NODE AT $(0, 0)$

Since the nodes are points of a homogeneous Poisson process, it follows that the location of any node that is known to be in $B'(0, 0; r_0)$ must be uniformly distributed over $B'(0, 0; r_0)$. In other words, the (unconditional) cdf of its distance R from $(0, 0)$ is given by

$$f_R(r) = 2r/r_0^2, \quad 0 \leq r \leq r_0.$$

If the shadow fade between this node and a node at $(0, 0)$ is $10^{Z/10}$, the probability that there is a direct connection between these nodes is (assuming r_0 very large)

$$\begin{aligned} &\mathbb{P}\{R < r_{\mathrm{n}} \exp(hZ/\delta)\} \\ &= \int_{-\infty}^{\infty} \frac{\exp[-z^2/(2\sigma^2)]}{\sigma\sqrt{2\pi}} \int_0^{\min\{r_0, r_{\mathrm{n}} \exp(hz/\delta)\}} \frac{2r}{r_0^2}\, \mathrm{d}r\, \mathrm{d}z \\ &= \frac{r_{\mathrm{n}}^2 \exp\left(\frac{2h^2\sigma^2}{\delta^2}\right)}{r_0^2} = \frac{\alpha}{r_0^2},\ \text{say.} \end{aligned} \quad (2)$$

Let N'_{n} be the random variable representing the number of nodes with which the node at $(0, 0)$ has a direct connection. Then we may write the probability mass function of N'_{n} as follows:

$$\begin{aligned} \mathbb{P}\{N'_{\mathrm{n}} = n\} &= \lim_{r_0 \to \infty} \mathrm{e}^{-\lambda_{\mathrm{n}} \pi r_0^2} \\ &\times \sum_{k=n}^{\infty} \frac{(\lambda_{\mathrm{n}} \pi r_0^2)^k}{k!} \binom{k}{n} \left(\frac{\alpha}{r_0^2}\right)^n \left(1 - \frac{\alpha}{r_0^2}\right)^{k-n}, \end{aligned} \quad (3)$$

where the summation over k starts at $k = n$ because k, the number of nodes within a circle of radius r_0, cannot be less than n, the number of nodes within r_0 that have a connection to the node at $(0, 0)$. Defining $m = k - n$, we can rewrite (3) as follows:

$$\begin{aligned} &\mathbb{P}\{N'_{\mathrm{n}} = n\} \\ &= \lim_{r_0 \to \infty} \frac{(\lambda_{\mathrm{n}} \pi \alpha)^n}{\mathrm{e}^{\lambda_{\mathrm{n}} \pi r_0^2} n!} \sum_{m=0}^{\infty} \frac{1}{m!} \left[(\lambda_{\mathrm{n}} \pi r_0^2) \left(1 - \frac{\alpha}{r_0^2}\right)\right]^m \\ &= \exp(-\lambda_{\mathrm{n}} \pi \alpha) \frac{(\lambda_{\mathrm{n}} \pi \alpha)^n}{n!}, \quad n = 0, 1, \ldots \\ &\Leftrightarrow\ N'_{\mathrm{n}} \sim \mathrm{Poiss}\left(\lambda_{\mathrm{n}} \pi r_{\mathrm{n}}^2 \exp\left(\frac{2h^2\sigma^2}{\delta^2}\right)\right), \end{aligned} \quad (4)$$

i.e., the number of "neighbors" of an arbitrary node is Poisson with mean given by $\lambda_{\mathrm{n}}\pi r_{\mathrm{n}}^2 \exp(2h^2\sigma^2/\delta^2)$. In particular, the probability that the node at $(0,0)$ has no connection to any other node is

$$q_{\mathrm{n}} = \mathbb{P}\{N'_{\mathrm{n}} = 0\} = \exp\left[-\lambda_{\mathrm{n}}\pi r_{\mathrm{n}}^2 \exp(2h^2\sigma^2/\delta^2)\right]. \tag{5}$$

VI. PROBABILITY THAT A NODE AT $(0,0)$ CANNOT COMMUNICATE WITH ANOTHER NODE AT A DISTANCE D AWAY IN t OR FEWER HOPS

Without loss of generality, let the other node be located at $(D,0)$. Let us call the probability of interest $q_t(D)$. It is clear that a connection cannot be established between the node at $(0,0)$ and the node at $(D,0)$ in $\leq t$ hops if and only if there is no direct connection between these two nodes, and any node that has a direct connection to one of these nodes, say, the node at $(0,0)$, cannot communicate with the other node, i.e., the node at $(D,0)$, in $\leq t-1$ hops.

Note that $q_1(D)$, the probability that there is no direct connection between the nodes at $(0,0)$ and $(D,0)$, is simply given by

$$q_1(D) = \mathbb{P}\left\{D > r_{\mathrm{n}}\mathrm{e}^{hZ/\delta}\right\} = 1 - Q\left(\frac{\ln(D/r_{\mathrm{n}})}{h\sigma/\delta}\right), \tag{6}$$

where $10^{Z/10}$ is the shadow fade between these two nodes, and $Q(x) = \int_x^\infty \exp(-u^2/2)\,\mathrm{d}u/\sqrt{2\pi}$. Note that $q_1(D)$ is only a function of the normalized distance D/r_{n} and the parameter $h\sigma/\delta$ and may be written as

$$q_1(D) = \tilde{q}_1\left(\frac{D}{r_{\mathrm{n}}}, \frac{h\sigma}{\delta}\right), \quad \tilde{q}_1(x,a) \equiv 1 - Q\left(\frac{\ln x}{a}\right). \tag{7}$$

If a node which has a direct connection to the node at $(0,0)$ is located at the point with polar coordinates (R,Θ), then the distance from this location to the node at $(D,0)$ is $\sqrt{D^2 + R^2 - 2DR\cos\Theta}$, and the probability that no connection exists between these two nodes in $\leq t-1$ steps is $q_{t-1}(\sqrt{D^2 + R^2 - 2DR\cos\Theta})$.

Since Θ is uniformly distributed over $[0,2\pi)$ and independent of R, we see that the joint pdf of (R,Θ) under the condition that a direct connection exists between this point and $(0,0)$ is given by

$$\begin{aligned}
\tilde{f}_{R,\Theta}(r,\theta) &\equiv f_{R,\Theta|R<r_{\mathrm{n}}\exp(hZ/\delta)}(r,\theta) \\
&= f_\Theta(\theta) f_{R|R<r_{\mathrm{n}}\exp(hZ/\delta)}(r) \\
&= \frac{1}{2\pi}\frac{\mathrm{d}}{\mathrm{d}r}\mathbb{P}\{R \leq r | R < r_{\mathrm{n}}\exp(hZ/\delta)\} \\
&= \frac{1}{2\pi\mathbb{P}\{R < r_{\mathrm{n}}\mathrm{e}^{hZ/\delta}\}}\frac{\mathrm{d}}{\mathrm{d}r}\mathbb{P}\{R \leq r,\, R < r_{\mathrm{n}}\exp(hZ/\delta)\} \\
&= \frac{1}{2\pi\mathbb{P}\{R < r_{\mathrm{n}}\mathrm{e}^{hZ/\delta}\}}\frac{\mathrm{d}}{\mathrm{d}r}\mathbb{P}\left\{R \leq r,\, Z > \frac{\ln(R/r_{\mathrm{n}})}{h/\delta}\right\} \\
&= \frac{r_0^2}{2\pi r_{\mathrm{n}}^2 \exp\left(\frac{2h^2\sigma^2}{\delta^2}\right)}\frac{\mathrm{d}}{\mathrm{d}r}\int_0^r \frac{2r'}{r_0^2} Q\left(\frac{\ln(r'/r_{\mathrm{n}})}{h/\delta}\right)\mathrm{d}r' \\
&= \frac{1}{2\pi}\,\frac{2r}{r_{\mathrm{n}}^2\exp(2h^2\sigma^2/\delta^2)} Q\left(\frac{\ln(r/r_{\mathrm{n}})}{h\sigma/\delta}\right).
\end{aligned} \tag{8}$$

The probability that there is no connection between the nodes at $(0,0)$ and $(D,0)$ in t hops or less is the probability that (a) there is no direct connection between these two nodes, and (b) that either (i) the node at $(0,0)$, say, has no direct connection to any other node at all, or (ii) any other node to which it does have a direct connection in turn does not have a connection to the node at $(D,0)$ in $t-1$ hops or less. Thus we may write the following recursive formula for $q_t(D)$:

$$\begin{aligned}
&q_t(D) \\
&= q_1(D)\Bigg[\mathbb{P}\{\text{Node at } (0,0) \text{ has no direct connection to} \\
&\qquad \text{any other node that is not at } (D,0)\} + \sum_{k=1}^{\infty}\mathbb{P}\{N'_{\mathrm{n}} = k\} \\
&\qquad \times \mathbb{P}\{\text{None of these } k \text{ nodes can communicate} \\
&\qquad \text{with the node at } (D,0) \text{ in } \leq t-1 \text{ hops} \,\Big|\, \text{these} \\
&\qquad \text{nodes can communicate directly with the node at } (0,0)\}\Bigg] \\
&= q_1(D)\Bigg[\mathbb{P}\{N'_{\mathrm{n}} = 0\} + \sum_{k=1}^{\infty}\mathbb{P}\{N'_{\mathrm{n}} = k\} \\
&\qquad \times \mathbb{P}\{\text{An arbitrary one of these nodes cannot communicate} \\
&\qquad \text{with the node at } (D,0) \text{ in } \leq t-1 \text{ hops} \,\Big|\, \text{this node} \\
&\qquad \text{can communicate directly with the node at } (0,0)\}^k\Bigg] \\
&= q_1(D)\exp(-\lambda_{\mathrm{n}}\pi\alpha) \\
&\qquad \times \left\{1 + \sum_{k=1}^{\infty}\frac{(\lambda_{\mathrm{n}}\pi\alpha)^k}{k!}\left[\int_0^\infty\int_0^{2\pi}\tilde{f}_{R,\Theta}(r,\theta)\right.\right. \\
&\qquad\quad \left.\left. \times\, q_{t-1}(\sqrt{D^2 + r^2 - 2Dr\cos\theta})\,\mathrm{d}\theta\,\mathrm{d}r\right]^k\right\} \\
&= q_1(D)\exp\left\{-\lambda_{\mathrm{n}}\pi\alpha\left[1 - \int_0^\infty\int_0^{2\pi}\tilde{f}_{R,\Theta}(r,\theta)\right.\right. \\
&\qquad\quad \left.\left. \times\, q_{t-1}(\sqrt{D^2 + r^2 - 2Dr\cos\theta})\,\mathrm{d}\theta\,\mathrm{d}r\right]\right\} \\
&= q_1(D)\exp\left\{-\lambda_{\mathrm{n}}\pi r_{\mathrm{n}}^2\mathrm{e}^{2h^2\sigma^2/\delta^2}\right. \\
&\qquad \times \left[1 - \int_0^\infty\int_0^{2\pi}\frac{r}{\pi r_{\mathrm{n}}^2\mathrm{e}^{2h^2\sigma^2/\delta^2}} Q\left(\frac{\ln(r/r_{\mathrm{n}})}{h\sigma/\delta}\right)\right. \\
&\qquad\quad \left.\left. \times\, q_{t-1}(\sqrt{D^2 + r^2 - 2Dr\cos\theta})\,\mathrm{d}\theta\,\mathrm{d}r\right]\right\},
\end{aligned} \tag{9}$$

where we assume that the events that two distinct nodes, each with a direct connection to the node at $(0,0)$, cannot communicate with the node at $(D,0)$ in exactly $t-1$ hops are independent. Clearly, while this assumption is true for $t = 2$, it does not hold for $t > 2$, and $q_t(D)$ as given by (9) therefore provides only a lower bound on the t-hop outage probability for $t > 2$. In the sequel, we distinguish between exact outage probability values, denoted $q_t(D)$, and the lower bound given by the right hand side of (9), denoted $\hat{q}_t(D)$. (Chandler implicitly makes a similar independence assumption, so his final expression [4, eq. (10)] is also a lower bound.)

Thus $\hat{q}_t(D)$ for arbitrary t can be obtained by recursion using (9) and (6). Further, if we define

$$\tilde{D} = D/r_{\mathrm{n}}, \quad a = h\sigma/\delta, \quad \tilde{\lambda}_{\mathrm{n}} = \lambda_{\mathrm{n}} r_{\mathrm{n}}^2, \quad \tilde{r} = r/r_{\mathrm{n}},$$

then we may rewrite (9) as follows:

$$\hat{q}_t(D) = \tilde{q}_1(\tilde{D}, a) \exp\Bigg\{ - \tilde{\lambda}_{\mathrm{n}} \pi \mathrm{e}^{2a^2} \Bigg[1 - \frac{1}{\pi \mathrm{e}^{2a^2}} \times \int_0^\infty \int_0^{2\pi} \tilde{r} Q\left(\frac{\ln \tilde{r}}{a}\right) \hat{q}_{t-1}(r_{\mathrm{n}} \sqrt{\tilde{D}^2 + \tilde{r}^2 - 2\tilde{D}\tilde{r}\cos\theta})\,\mathrm{d}\theta\,\mathrm{d}\tilde{r} \Bigg] \Bigg\}.$$

From this expression and (7), it follows by mathematical induction that for all $t \geq 1$, $\hat{q}_t(D)$ is a function only of the parameter $h\sigma/\delta$, the normalized distance D/r_{n}, and the normalized intensity $\lambda_{\mathrm{n}} r_{\mathrm{n}}^2$:

$$\hat{q}_t(D) \equiv \tilde{q}_t\left(\frac{D}{r_{\mathrm{n}}}, \frac{h\sigma}{\delta}, \lambda_{\mathrm{n}} r_{\mathrm{n}}^2\right), \tag{10}$$

where

$$\begin{aligned} \tilde{q}_t(\tilde{D}, a, \tilde{\lambda}_{\mathrm{n}}) &= \tilde{q}_1(\tilde{D}, a) \\ &\times \exp\Bigg\{ - \tilde{\lambda}_{\mathrm{n}} \pi \mathrm{e}^{2a^2} \Bigg[1 - \int_0^\infty \int_0^{2\pi} \frac{\tilde{r}}{\pi \mathrm{e}^{2a^2}} Q\left(\frac{\ln \tilde{r}}{a}\right) \\ &\times \tilde{q}_{t-1}(\sqrt{\tilde{D}^2 + \tilde{r}^2 - 2\tilde{D}\tilde{r}\cos\theta}, a, \tilde{\lambda}_{\mathrm{n}})\,\mathrm{d}\theta\,\mathrm{d}\tilde{r} \Bigg] \Bigg\}. \end{aligned} \tag{11}$$

Thus a table or chart of the values of $\tilde{q}_t(\cdot, \cdot, \cdot)$, which need be computed only once, is *universally applicable*.

In particular, it follows after some algebra that

$$\begin{aligned} q_2(D) &= q_1(D) \exp\Bigg\{ - \lambda_{\mathrm{n}} \pi \alpha \Bigg[1 - \int_0^\infty \mathrm{d}r \int_0^{2\pi} \mathrm{d}\theta \\ &\quad \times \tilde{f}_{R,\Theta}(r, \theta) q_1(\sqrt{D^2 + r^2 - 2Dr\cos\theta}) \Bigg] \Bigg\} \\ &= \tilde{q}_1(\tilde{D}, a) \exp\Bigg\{ - \tilde{\lambda}_{\mathrm{n}} \int_0^\infty \int_0^{2\pi} \tilde{r} Q\left(\frac{\ln \tilde{r}}{a}\right) \\ &\quad \times Q\left(\frac{\ln \sqrt{\tilde{D}^2 + \tilde{r}^2 - 2\tilde{D}\tilde{r}\cos\theta}}{a}\right) \mathrm{d}\theta\,\mathrm{d}\tilde{r} \Bigg\} \end{aligned} \tag{12}$$

Note that this is the *exact* 2-hop outage probability.

An alternative derivation of $q_2(D)$ yielding a more intuitive and slightly simpler expression than (12) is as follows: from the definition of $q_2(D)$, it is clear that the term multiplying $q_1(D)$ on the right hand side of (12) must be the probability that there is no connection between the nodes at $(0,0)$ and $(D,0)$ in *exactly* 2 hops. In other words, any node that has a connection to the node at $(0,0)$, say, must not have a connection to the other node at $(D,0)$. Let us label the shadow fades on the link between an arbitrary node and the two given nodes at $(0,0)$ and $(D,0)$ by Z and Z' respectively. For any given values of Z and Z', this node has a connection to both the nodes at $(0,0)$ and $(D,0)$ if and only if it lies in the region of intersection of the disks centered at $(0,0)$ and $(D,0)$ with radii $r_{\mathrm{n}} \mathrm{e}^{hZ/\delta}$ and $r_{\mathrm{n}} \mathrm{e}^{hZ'/\delta}$ respectively. Equivalently, a two-hop connection between the nodes at $(0,0)$ and $(D,0)$ does not exist if and only if there is no node in the region of intersection of the disks centered at $(0,0)$ and $(D,0)$ with radii $r_{\mathrm{n}} \mathrm{e}^{hZ/\delta}$ and $r_{\mathrm{n}} \mathrm{e}^{hZ'/\delta}$ respectively. Then we have:

$$\begin{aligned} &q_2(D) \\ &= q_1(D) \lim_{r_0 \to \infty} \sum_{k=0}^{\infty} \mathbb{P}\{N_{\mathrm{n}}(B'(0,0;r_0) \smallsetminus \{(D,0)\}) = k\} \\ &\quad \times \mathbb{P}\{\text{node at } (0,0) \text{ cannot communicate with } (D,0) \\ &\quad \text{in } \leq 2 \text{ hops} \mid N_{\mathrm{n}}(B'(0,0;r_0) \smallsetminus \{(D,0)\}) = k\} \\ &= q_1(D) \lim_{r_0 \to \infty} \sum_{k=0}^{\infty} \exp(-\lambda_{\mathrm{n}} \pi r_0^2) \frac{(\lambda_{\mathrm{n}} \pi r_0^2)^k}{k!} \\ &\quad \times [1 - \mathbb{P}\{\text{An arbitrary node in } B'(0,0;r_0) \smallsetminus \{(D,0)\} \\ &\quad \text{has a connection to both } (0,0) \text{ and } (D,0) | \text{ node} \\ &\quad \text{location uniform in } B(0,0;r_0)\}]^k, \end{aligned} \tag{13}$$

where we use the independence of the events that the different nodes in $B'(0,0;r_0) \smallsetminus \{(D,0)\}$ have connections to both $(0,0)$ and $(D,0)$.

Summing the infinite series in (13), we can rewrite it as

$$\begin{aligned} &q_2(D) \\ &= q_1(D) \lim_{r_0 \to \infty} \exp(-\lambda_{\mathrm{n}} \pi r_0^2) \exp[\lambda_{\mathrm{n}} \pi r_0^2 (1 - \mathbb{P}\{\text{An arbitrary} \\ &\quad \text{node in } B'(0,0;r_0) \smallsetminus \{(D,0)\} \text{ has} \\ &\quad \text{a connection to both } (0,0) \text{ and } (D,0) | \text{ node location} \\ &\quad \text{uniform in } B(0,0;r_0)\})] \\ &= q_1(D) \lim_{r_0 \to \infty} \exp[-\lambda_{\mathrm{n}} \pi r_0^2 \, \mathbb{P}\{\text{An arbitrary node} \\ &\quad \text{in } B'(0,0;r_0) \smallsetminus \{(D,0)\} \text{ has a} \\ &\quad \text{connection to both } (0,0) \text{ and } (D,0) | \text{ node location} \\ &\quad \text{uniform in } B(0,0;r_0)\}]. \end{aligned} \tag{14}$$

Finally, since this node location is uniformly distributed on the disk $B(0,0;r_0)$, it follows from (1) that

$$\begin{aligned} &\mathbb{P}\{\text{An arbitrary node in } B'(0,0;r_0) \smallsetminus \{(D,0)\} \text{ has a} \\ &\quad \text{connection to both } (0,0) \text{ and } (D,0) | \text{ node location} \\ &\quad \text{uniform in } B(0,0;r_0)\} \\ &= \frac{\mathbb{E}_{Z,Z'}\left[\text{area}\left(B'(0,0;r_{\mathrm{n}} \mathrm{e}^{hZ/\delta}) \cap B'(D,0;r_{\mathrm{n}} \mathrm{e}^{hZ'/\delta})\right)\right]}{\text{area}(B(0,0;r_0))} \\ &= \frac{\mathbb{E}_{Z,Z'}\left[g\left(D; r_{\mathrm{n}} \mathrm{e}^{hZ/\delta}, r_{\mathrm{n}} \mathrm{e}^{hZ'/\delta}\right)\right]}{\pi r_0^2}, \end{aligned} \tag{15}$$

where [6, equation (11)]

$$\begin{aligned} g(r; a, b) &\equiv \text{area}\Big(B(x + r\cos\theta, y + r\sin\theta; a) \cap B(x, y; b)\Big) \\ &= \text{area}(B(r, 0; a) \cap B(0, 0; b)) \end{aligned}$$

$$
= \begin{cases} \pi(\min\{a,b\})^2, & r < |a-b|, \\ a^2 \cos^{-1}\left(\dfrac{r^2+a^2-b^2}{2ra}\right) & \\ +b^2 \cos^{-1}\left(\dfrac{r^2+b^2-a^2}{2rb}\right) & \\ -\dfrac{1}{2}\sqrt{(-r+a+b)(r+a-b)} & \\ \times\sqrt{(r-a+b)(r+a+b)}, & |a-b| \le r < a+b, \\ 0, & r \ge a+b. \end{cases} \tag{16}
$$

Substituting (15) into (14), we finally obtain

$$
\begin{aligned} & q_2(D) \\ &= q_1(D) \exp\left\{-\lambda_{\mathrm{n}} \mathbb{E}_{Z,Z'}\left[g\left(D; r_{\mathrm{n}} \mathrm{e}^{hZ/\delta}, r_{\mathrm{n}} \mathrm{e}^{hZ'/\delta}\right)\right]\right\} \\ &= q_1(D) \exp\left\{ - \frac{\lambda_{\mathrm{n}}}{2\pi} \int_{-\infty}^{\infty} \mathrm{e}^{-v^2/2} \int_{-\infty}^{\infty} \mathrm{e}^{-w^2/2} \right. \\ & \qquad \left. \times\, g(D; r_{\mathrm{n}} \mathrm{e}^{hw\sigma/\delta}, r_{\mathrm{n}} \mathrm{e}^{hv\sigma/\delta}) \,\mathrm{d}w \,\mathrm{d}v \right\}. \end{aligned} \tag{17}
$$

We now examine some interesting cases of practical relevance. Note that the network of nodes extends over the entire plane. However, we will now restrict the source and destination nodes to lie in a finite region, even though a path between that source and destination could go through other nodes that lie outside this region.

Region of interest is a finite disk: Suppose we are only interested in the case where the source and destination nodes lie in a disk of some large (but finite) radius R. The pdf of the distance D between two arbitrary nodes in this disk is known to be [7, p. 28]

$$
\begin{aligned} f_D(x) &= \frac{2x}{R^2}\left\{1 - \frac{2}{\pi}\left[\sin^{-1}\left(\frac{x}{2R}\right)\right.\right. \\ & \left.\left. + \frac{x}{2R}\sqrt{1-\left(\frac{x}{2R}\right)^2}\right]\right\}, 0 \le x \le 2R. \end{aligned} \tag{18}
$$

Region of interest is a finite rectangle: If the source and destination nodes are restricted to lie in a (large) rectangle with width w and height h, as is considered by Mullen [5] (say with $w < h$ without loss of generality), then the corresponding pdf of the distance D between two arbitrary nodes in this rectangle is [8, (2.10)]

$$
f_D(x) = \frac{4x}{hw} \begin{cases} \pi/2 - (1/w + 1/h)x & \\ +x^2/(2hw), & 0 \le x < w, \\ \sqrt{(x/w)^2 - 1} - (x/w + w/2h) & \\ + \sin^{-1}(w/x), & w \le x < h, \\ \sqrt{(x/w)^2-1} + \sqrt{(x/h)^2-1} & \\ - (x^2+h^2+w^2)/(2hw) & \\ + \sin^{-1}(w/x) - \cos^{-1}(h/x), & \\ & h \le x < \sqrt{w^2+h^2}. \end{cases}
$$

These pdfs can be substituted into (12) to calculate the probability Q_2 that a randomly chosen pair of nodes in the region of interest cannot communicate with 2 hops or less. For example, if the region of interest is a disk of radius R, we have

$$
Q_2 = \int_0^{2R} f_D(x) q_2(x) \,\mathrm{d}x.
$$

Define $u = R/r_{\mathrm{n}}, \quad \tilde{y} = x/(2R), \quad \tilde{x} = x/r_{\mathrm{n}} = 2u\tilde{y}$.
Then from (12) and (18), we may write Q_2 as follows:

$$
\begin{aligned} & Q_2(u) \\ &= \int_0^1 \tilde{f}(\tilde{y}) \tilde{q}_1(2u\tilde{y}, a) \exp\left\{ - \tilde{\lambda}_{\mathrm{n}} \int_0^{\infty} \int_0^{2\pi} \tilde{r} Q\left(\frac{\ln \tilde{r}}{a}\right) \right. \\ & \qquad \left. \times\, Q\left(\frac{\ln\sqrt{4u^2\tilde{y}^2 + \tilde{r}^2 - 4u\tilde{y}\tilde{r}\cos\theta}}{a}\right) \mathrm{d}\theta \,\mathrm{d}\tilde{r} \right\} \mathrm{d}\tilde{y}, \end{aligned} \tag{19}
$$

where

$$
\tilde{f}(\tilde{y}) = 8\tilde{y}\left\{1 - \frac{2}{\pi}[\sin^{-1}(\tilde{y}) + \tilde{y}\sqrt{1-(\tilde{y})^2}]\right\}, 0 \le \tilde{y} \le 1.
$$

VII. NUMERICAL RESULTS

In Fig. 1, we plot $q_2(D) \equiv \tilde{q}_2(D/r_{\mathrm{n}}, h\sigma/\delta, 1)$ as given by (12), versus D/r_{n} for $\delta = 4$, $\lambda_{\mathrm{n}} r_{\mathrm{n}}^2 = 1$, and $\sigma = 0, 2, 4, 8\,\mathrm{dB}$. We see that ignoring fading, i.e., assuming $\sigma = 0\,\mathrm{dB}$, as was done by many researchers, may lead to large errors. Fig. 2 compares $q_1(D) \equiv \tilde{q}_1(D/r_{\mathrm{n}}, h\sigma/\delta, 1)$ as given by (7) and $q_2(D) \equiv \tilde{q}_2(D/r_{\mathrm{n}}, h\sigma/\delta, 1)$ for the particular choice of $\sigma = 8\,\mathrm{dB}$, the other parameters remaining the same as above. In Fig. 3 we plot the probabilities $q_3(D)$ and $q_4(D)$ that a node at $(0,0)$ cannot connect to a node at $(D,0)$ in ≤ 3 or 4 hops respectively, as obtained from simulation for the same choice of parameters as in Fig. 2, versus D/r_{n}. As expected, we see that for large D/r_{n}, more hops are necessary to achieve a connection with high probability.

In Fig. 4, we plot the probability that a randomly-chosen pair of nodes in a disk of radius R cannot communicate in ≤ 2 hops, as given by (19), versus the ratio $u = R/r_{\mathrm{n}}$, for $\tilde{\lambda}_{\mathrm{n}} = 1$, $\sigma = 8\,\mathrm{dB}$, and $\delta = 4$.

VIII. CONCLUSIONS

We have studied an unbounded wireless ad hoc network, where the locations of the nodes follow a homogeneous Poisson process with intensity λ_{n}, and the radio propagation channels exhibit exponential attenuation with distance and independent lognormal fading. A fundamental property of such a network is the probability distribution of the minimal number of hops required to connect an arbitrary source node to a destination node at distance D. Alternatively, the problem can be stated in terms of $q_t(D)$ - the probability that a node cannot find a route to another node at distance D with up to t hops. This distribution has a significant impact on route discovery, throughput and power consumption. It serves as a convenient reference for evaluation of practical routing algorithms. We calculate $q_1(D)$ and $q_2(D)$ for arbitrary D, and derive a recursive formula yielding a lower bound for $q_t(D)$ when $t > 2$. We plot analytic results for $q_1(D)$ and $q_2(D)$, and

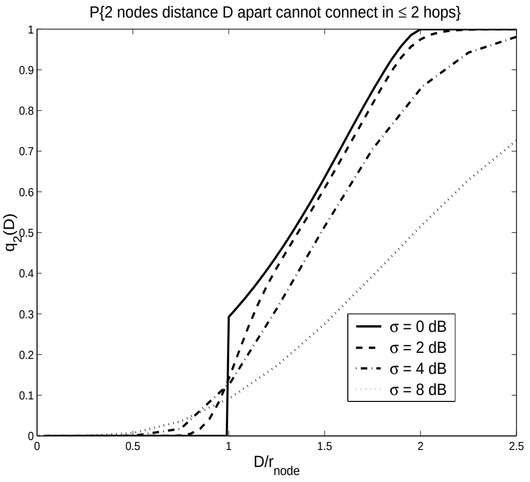

Fig. 1. Plot of $q_2(D)$ versus D for $\delta = 4$, $\lambda_{\rm n} r_{\rm n}^2 = 1$, and $\sigma = 0, 2, 4, 8\,{\rm dB}$.

results derived from simulations for $t > 2$. We show that the fading has a profound impact on $q_t(D)$, and ignoring it, which is common in the literature, may lead to biased results. As a possible application of our work, we quote known results for the pdf of D, the distance between a randomly-chosen pair of nodes in the network, for two cases of practical interest. This allows us to calculate the probability that a randomly chosen pair of nodes in a finite region cannot communicate in a given number of hops.

REFERENCES

[1] M. Zorzi, R. Rao, "Geographic Random Forwarding (GeRaF) for ad Hoc and Sensor Networks: Multihop Performance." *IEEE Transactions on Mobile Computing*, vol. 2, no. 4, pp. 337-348, Oct.-Dec. 2003.

[2] C. Bettstetter, J. Eberspaecher, "Hop Distances in Homogeneous Ad Hoc Networks." In *Proceedings of the IEEE Vehicular Technology Conference 2003-Spring*, vol. 4, pp. 2286-2290, Jeju Island, Republic of Korea, Apr. 2003.

[3] L.E. Miller, "Probability of a Two-Hop Connection in a Random Mobile Network." In *Proceedings of the Conference on Information Sciences and Systems 2001*, Baltimore, MD, USA, Mar. 2001.

[4] S.A.G. Chandler, "Calculation of Number of Relay Hops Required in Randomly Located Radio Network." *Electronics Letters*, vol. 25, no. 24, pp. 1669-1671, 23 Nov. 1989.

[5] J.P. Mullen, "Robust Approximations to the Distribution of Link Distances in a Wireless Network Occupying a Rectangular Region." *Mobile Computing and Communications Review*, vol. 7, no. 2, pp. 80-91, Apr. 2003.

[6] E. Weisstein, ed. *Mathworld.* http://mathworld.wolfram.com/Circle-CircleIntersection.html

[7] B.D. Ripley, *Statistical Inference for Spatial Processes*. Cambridge University Press, 1988.

[8] L.E. Miller, "Distribution of Link Distances in a Wireless Network." *Journal of Research of the National Institute of Standards and Technology*, vol. 106, no. 2, pp. 401-412, Mar.-Apr. 2001.

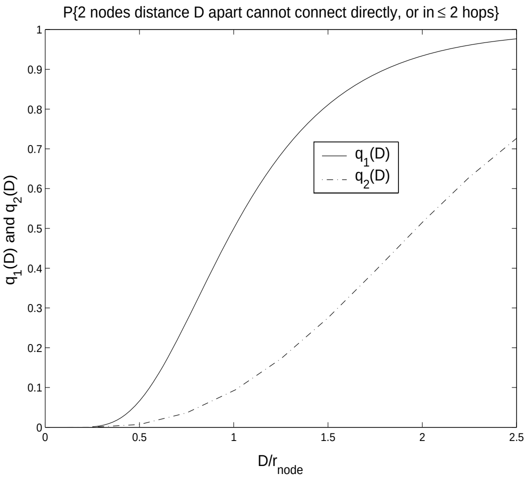

Fig. 2. Plot of $q_1(D)$ and $q_2(D)$ versus D for $\delta = 4$, $\sigma = 8\,{\rm dB}$, and $\lambda_{\rm n} r_{\rm n}^2 = 1$.

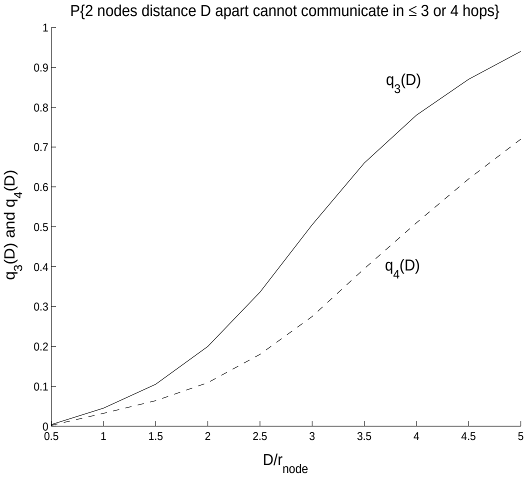

Fig. 3. Plot of $q_3(D)$ and $q_4(D)$ versus D as obtained from simulation for $\delta = 4$, $r_{\rm n} = 1\,{\rm km}$, $\lambda_{\rm n} = 1/r_{\rm n}^2 = 10^{-6}\,{\rm m}^{-2}$, and $\sigma = 8\,{\rm dB}$.

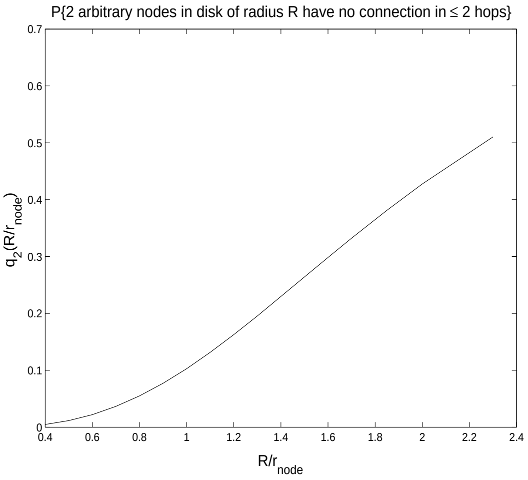

Fig. 4. Plot of $Q_2(u)$ versus $u = R/r_{\rm n}$ as given by (19) for $\delta = 4$, $\tilde{\lambda}_{\rm n} = 1$, and $\sigma = 8\,{\rm dB}$.