A Hierarchical Model for a Sensor Network

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Abstract

This paper introduces a new model for a sensor network in which sensors report information to supervisors through cluster heads which are themselves selected at random from the sensors. Assuming an inverse power law for attenuation, various shadowing or fading models, and uniform random spatial distributions of sensors and supervisors, the probability distribution of the number of sensors reporting to a supervisor is obtained. The result is extended to the case of a hierarchy of supervisors, which may or may not themselves be sensors, each reporting to the next level up the hierarchy.

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1 Introduction

Sensor networks are currently generating a large literature: an overview can be found in [1, 2], while [3] considers uniform spatial distributions of the kind appearing in this paper, which extends the analysis of [4] to a sensor network. Sensors, distributed at random over an infinite plane, are assigned to clusters and report when required to a cluster head which in turn reports to one of a number of supervisors. Cluster heads are chosen from the sensors by a self-election process triggered by a call from the supervisors, and all sensors have the same probability of election. This election process is repeated at intervals to share the extra power burden involved in being a cluster head.
Supervisors are also distributed uniformly over the infinite plane. When the self-election process is triggered each randomly elected cluster head will report to the supervisor from which it receives the strongest signal. Non-elected sensors will likewise report to the cluster head from which they receive the strongest signal. Thus sensors always report to the same supervisor when they act as cluster heads, but may report to cluster heads reporting to different supervisors when they are acting purely as sensors.
Because of the random self-election procedure, cluster heads are also randomly distributed on the infinite plane. Thus we have a three-level hierarchy of nodes, similar to the conventional two-level hierarchy of mobiles and base stations, with sensors acting as mobiles, cluster heads as base stations, and supervisors as a kind of super base station. The random distribution at all levels means that the form of statistical analysis adopted in [4] can be used here.
The simplest model analysed there assumes an inverse power law for attenuation and lognormal shadowing. The power loss in decibels at (random) distance R from a node is

$$L = k_0 + k_1 \ln R + S, \tag{1}$$

where k_0 and k_1 are constants, and S is a shadowing effect, normally distributed with mean zero, variance σ^2. The condition for audibility is $L \leq l_1$, a constant.

2 Coverage

It may be deduced from [4] that, assuming a uniform distribution of cluster heads, the number of cluster heads audible to a given sensor has a Poisson distribution with mean

$$\mu_c = \pi \rho_c e^{2(\sigma^2/k_1^2 - k_0/k_1)} e^{2l_1/k_1}, \tag{2}$$

where ρ_c is the cluster head density. Much of the following analysis will be based on this and similar means, so it should be noted immediately that its value is, at a cost, under the control of the system manager. In particular, it can be increased by increasing either ρ_c or l_1 (achievable by increasing either the power of the transmitted signal or the sensitivity of the receiver). A large mean is desirable because the probability that the sensor is audible to no cluster head is $e^{-\mu_c}$, which decreases as μ_c increases. In particular, it is about 5 percent for a mean of 3, decreasing below 2 percent for 4 and 1 percent for 5.
Values for μ_c can be obtained from [4] for a two-slope model (for example one with a change from an inverse square law to an inverse fourth power one at some specified distance) and for models with hot spots, and are also available for models with Rayleigh and Ricean fading as well as for the so-called Suzuki distribution. In all cases the distributions are Poisson.
For our sensor network model we assume a density ρ_s for individual sensors and a probability of self-election as a cluster head p_e. The density of cluster heads is $\rho_c = p_e \rho_s$ and that of non-elected sensors is $(1-p_e)\rho_s$,

and both of these are uniformly (and independently) distributed. Substituting these densities in (2) or the result corresponding to any other model mentioned above, $\mu_{c(s)}$, the mean number of cluster heads audible to a given (non-elected) sensor, and $\mu_{s(c)}$, the mean number of (non-elected) sensors audible to a given cluster head are obtained. It follows immediately that

$$\mu_{s(c)}/\mu_{c(s)} = (1-p_e)/p_e. \tag{3}$$

The probability that a given non-elected sensor is audible to no cluster head is $e^{-\mu_{s(c)}}$, and this is also the probability that it is not audible to any supervisor, since the selection procedure for cluster heads ensures that these are all audible to a supervisor.
Some sensors may not be audible to any supervisor: this does not necessarily imply that the sensor is never within range of a cluster head, but it does mean that the sensor does not use up its quota of power as a cluster head. The analysis above assumes the proportion of such sensors is negligible: if this is not so, then p_e must be modified accordingly.

3 Load

Suppose that cluster head C_0 can hear M sensors, and that a given one of these can hear $J+1$ cluster heads. Then the probability that $J=j$ is equal to $\mu_{c(s)}^j e^{-\mu_{c(s)}}/j!$, and the conditional probability, given j, that C_0 is the cluster head receiving the strongest signal from this sensor is $1/(j+1)$. Multiplying these probabilities and summing over j, the probablility that C_0 is the cluster head receiving the strongest signal from the given sensor is $p=(1-e^{-\mu_{c(s)}})/\mu_{c(s)}$. This is very close to $1/\mu_{c(s)}$, the intuitively expected result: the term $1-e^{-\mu_{c(s)}}$ arises from the sensors out of range of any cluster head, and is very close to 1 for any practically useful scenario. Let $q=1-p$, and define a probability generating function $ps+q$ for the given sensor. Then the probability generating function for the number of sensors reporting to C_0, given $M=m$, is $(ps+q)^m$. Hence the probability generating function for the number of sensors reporting to C_0 is

$$\sum_{m=0}^{\infty} \frac{\mu_{s(c)}^m e^{-\mu_{s(c)}}}{m!}(ps+q)^m,$$

which is $e^{\mu_{s(c)}p(s-1)}$, and this is the probability generating function of a Poisson distribution with mean $\mu_{s(c)}p$, a result which might have been expected.
Denote this mean by μ_1 and let N_1 be the number of individual sensors reporting to a given cluster head and N_2 the number of cluster heads reporting to a given supervisor, which will have a Poisson distribution with mean μ_2 obtained by replacing $\mu_{c(s)}$ and $\mu_{s(c)}$ above by $\mu_{S(c)}$, the mean number of supervisors audible to a given cluster head, and $\mu_{c(S)}$, the mean number of cluster heads audible to a given supervisor, respectively.
For general μ_1 and μ_2 and given $N_2=n_2$, the total number (N) of non-elected sensors reporting (through cluster heads) to the given supervisor has a Poisson distribution with mean $n_2\mu_1$, and the probability generating function $\Pi_{N|N_2=n_2}(s)$ is therefore $e^{n_2\mu_1(s-1)}$. Multiplying by the Poisson probability for n_2 (and by $s_2^{n_2}$) and summing, the unconditional joint probability generating function for N_2 and N is

$$\Pi_{N_2,N}(s_2,s) = e^{\mu_2[s_2 e^{\mu_1(s-1)}-1]}. \tag{4}$$

Putting $s_2=1$, the generating function for the number of non-elected sensors is

$$e^{\mu_2[e^{\mu_1(s-1)}-1]}, \tag{5}$$

and differentiating this with respect to s and setting $s=1$ in the derivative, the mean number of non-elected sensors reporting to the supervisor is $\mu_1\mu_2$, as must be expected. Putting $s_2=s$ in (4), the generating function for the total number of sensors, elected or not, becomes

$$e^{\mu_2[se^{\mu_1(s-1)}-1]}, \tag{6}$$

and the mean is now $(\mu_1+1)\mu_2$.
A second differentiation leads to the variance. In the case of (6) it is $(\mu_1+1)^2\mu_2+\mu_1\mu_2$.

4 Higher Level Hierarchy

The model can be extended to introduce one or more levels of assistant supervisors between cluster heads and supervisors, either self-elected from the sensors, or distinct. They will be categorised according to their level in the hierarchy - to be precise, a supervisor at level n will report to one at level $n+1$ information reported to it by supervisors at level $n-1$, and in addition its own information if it is itself a sensor. Cluster heads will occupy level 1, and individual sensors level zero.
Let μ_n be the mean number of supervisors at level $n-1$ reporting to a given supervisor at level n, and let $\Pi_n(s)$ be the probability generating function for the number of sensors whose information reaches a given supervisor at level n. Then the reasoning used in obtaining (5) and (6) gives

$$\Pi_n(s) = e^{\mu_n[\Pi_{n-1}(s)-1]} \tag{7}$$

in the case when level $n-1$ supervisors are not sensors. If the level $n-1$ supervisors are sensors, then

$$\Pi_n(s) = e^{\mu_n[s\Pi_{n-1}(s)-1]}. \tag{8}$$

The initial condition is $\Pi_1(s) = e^{\mu_1(s-1)}$.
Means and recurrence formulae for individual probabilities are now obtained by differentiation. From (7),

$$\ln \Pi_n(s) = \mu_n[\Pi_{n-1}(s) - 1], \quad (9)$$

so, differentiating and rearranging,

$$\frac{\partial \Pi_n}{\partial s} = \mu_n \Pi_n \frac{\partial \Pi_{n-1}}{\partial s}. \quad (10)$$

Setting $s = 1$, $\frac{\partial \Pi_n}{\partial s}$ is the mean number of sensors whose information reaches a given level n supervisor, and is therefore, by repeated application of (10), equal to $\prod_{i=1}^{n} \mu_i$ in the case where supervisors are not sensors, but μ_i in this product becomes $\mu_i + 1$ when level $i-1$ $(i > 1)$ supervisors are sensors, for then, from (8),

$$\ln \Pi_n(s) = \mu_n[s\Pi_{n-1}(s) - 1], \quad (11)$$

and

$$\frac{\partial \Pi_n}{\partial s} = \mu_n \Pi_n \left[s\frac{\partial \Pi_{n-1}}{\partial s} + \Pi_{n-1} \right]. \quad (12)$$

For individual probabilities, differentiate (10) r times by Leibniz's Theorem.

$$\frac{\partial^{r+1}\Pi_n}{\partial s^{r+1}} = \mu_n \sum_{u=0}^{r} \frac{r!}{u!(r-u)!} \frac{\partial^u \Pi_n}{\partial s^u} \frac{\partial^{r-u+1}\Pi_{n-1}}{\partial s^{r-u+1}}. \quad (13)$$

Dividing both sides by $(r+1)!$ and setting $s = 0$, the equation may be written

$$p_{n,r+1} = \frac{\mu_n}{r+1} \sum_{u=0}^{r} (r-u+1) p_{n,u} p_{n-1,r-u+1}, \quad (14)$$

where $p_{n,r}$, the probability that there are exactly r sensors reporting to a given level n supervisor, is the coefficient of s^r in $\Pi_n(s)$. This result applies when level $n-1$ supervisors are not sensors. If they are, then Leibniz's Theorem is applied to (12), and setting $s = 0$ leads to the slightly different result

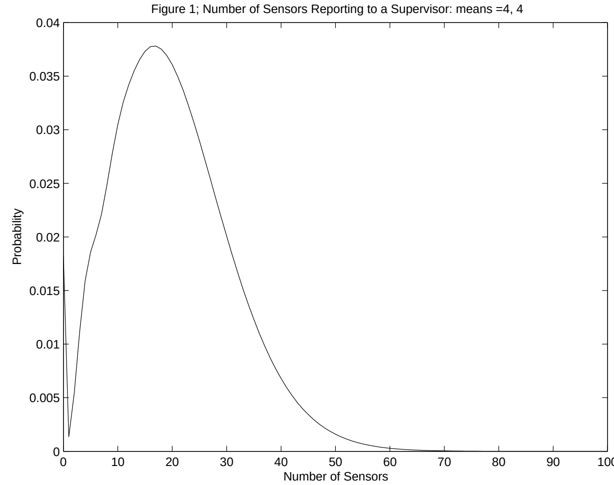
$$p_{n,r+1} = \frac{\mu_n}{r+1} \sum_{u=0}^{r} (r-u+1) p_{n,u} p_{n-1,r-u}. \quad (15)$$

This gives a means of finding probabilities recursively, illustrated and discussed in the next section.

5 Examples

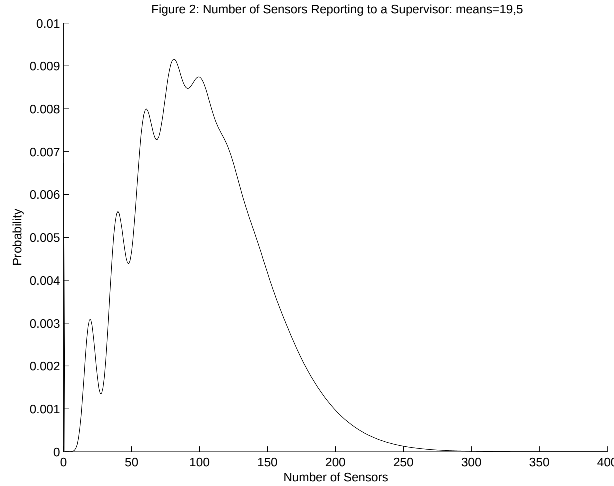
The probability distribution of the load has a number of interesting and unexpected features. The first is illustrated with an unrealistically high probability of self-election, corresponding to $\mu_1 = 4$, for which p_e is very close to 0.2, depending on the mean number of non-elected sensors reporting to a cluster head, as explained in section 3. Here, and in future, we use the approximation there described to obtain a distribution effectively independent of this mean number of sensors. We also take $\mu_2 = 4$. The distribution is shown in Figure 1. Note that it is a discrete distribution. The mysterious dip at the beginning of the plot is predicted by the equations: the first two probabilities are $e^{-\mu_2}$ and $\mu_2 e^{-\mu_1} e^{-\mu_2}$. Thus the only circumstance in which the number of sensors reporting can be 1 is that only 1 cluster head is reporting and no sensors are attached to it, a very unlikely situation.

Figure 1; Number of Sensors Reporting to a Supervisor: means =4, 4

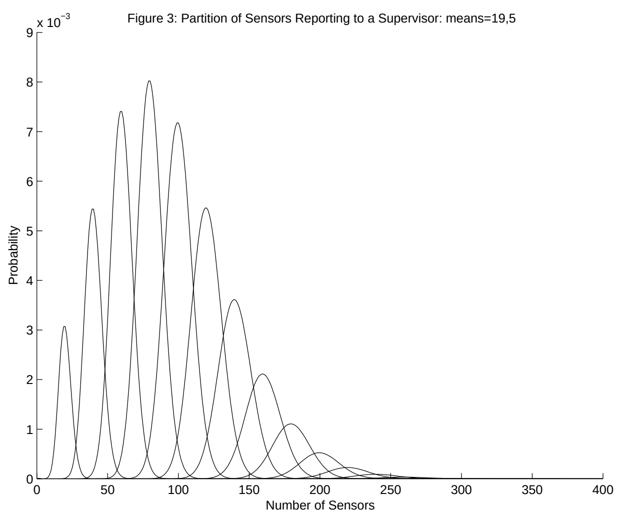


For Figure 2 μ_1 is taken as 19, giving $p_e = 0.05$ approximately, and μ_2 is taken as 5. The plot shows an unexpected series of bumps, which might initially cast doubt on the numerical accuracy of the procedure. This is a serious concern, since in the recurrence formula (15) each probability depends on the previously calculated ones, hence beginning with very small values, so errors are likely to accumulate.

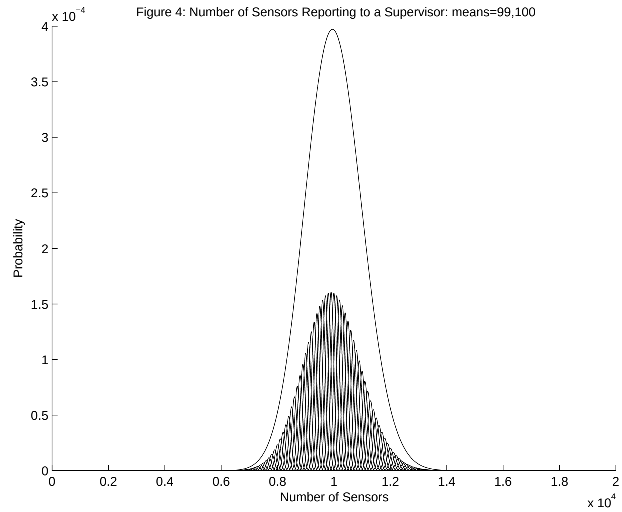
Figure 2: Number of Sensors Reporting to a Supervisor: means=19,5



However, the mean and variance of the calculated distribution agree remarkably well with the correct values, so a physical explanation must be sought, and this is forthcoming if we return to the derivation of the probability generating function in section 3, where it was observed that the total number of non-elected sensors reporting to a specified number of cluster heads has a Poisson distribution. Figure 3 shows the distribution of Figure 2 partitioned into contributions from 1, 2, 3, ... cluster heads, and it is immediately apparent that these contributions do combine to give the plot of Figure 2. The individual plots are essentially Poisson distributions, but scaled by the Poisson probability of the corresponding number of cluster heads and shifted to the right by that same munber. Thus the one with the highest peak corresponds to 4 cluster heads and is therefore a Poisson distribution with mean 76 (4×19), shifted 4 units to the right to include the 4 cluster heads in the count, and with all the probabilities multiplied by $5^4e^{-5}/4!$.



This second procedure, with the combined probability obtained by adding the probabilities for the partitions, is also much quicker than the recurrence formula, which rapidly demands a prohibitive amount of computing time as the means increase. In addition, probabilities can be calculated outwards from the large values in the middle of the distributions, hence virtually eliminating concerns about numerical accuracy. The final figure, Figure 4, is obtained in this way, and shows both the partition and the combined probability distribution for the realistic situation in which one percent of sensors are self-elected as cluster heads (and therefore taking $\mu_1 = 99$), and the mean number of cluster heads reporting to a supervisor is 100 ($\mu_2 = 100$). The distribution is very close to normality: when plotted, the normal curve is indistinguishable from the actual distribution.



6 Conclusions

The new model gives a probability distribution for the number of sensors reporting through cluster heads to a supervisor, and this distribution is close to normality for realistic parameter values. The distribution depends almost entirely on the proportion of sensors acting as cluster heads and the mean number of cluster heads reporting to a given supervisor.

References

[1] I. F. Akyildiz, W. Su, Y. Sankarasubramaniam, E. Cayirci, *"A Survey on Sensor Networks"*, IEEE Communications Magazine, Aug. 2002, 102-114.

[2] M. Tubaishat, S. Madria, *"Sensor Networks: an Overview"*, IEEE Potentials, Vol. 22 Issue 2, April/May 2003.

[3] C. Bettstetter, *"On the Connectivity of Wireless Multihop Networks with Homogeneous and Inhomogeneous Range Assignment,"* Vehicular Technology Conference, 2002, Proceedings, VTC 2002-Fall, Volume 3, 24-28 Sept. 2002, 1706-1710.

[4] J. Orriss, S. K. Barton, *"Probability Distributions for the Number of Radio Transceivers Which Can Communicate with One Another"*, IEEE Transactions on Communications, Vol. 51 No. 4, Apr. 2003, 676-681.